SAHLQVIST-VAN BENTHEM ALGORITHM

BASED ON THE NOTES BY IAN HODKINSON AND SECTIONS 3.5-3.6 IN BRV

A *boxed atom* is a modal formula of the form $\Box^n p$, for some $n \in \mathbb{N}$, where p is a propositional variable, and $\Box^n p$ is defined by the rule: $\Box^0 p = p$, $\Box^1 p = \Box p$, $\Box^{n+1} p = \Box(\Box^n p)$, $n \in \mathbb{N}$.

A *simple Sahlqvist antecedent* is built from $\bot, \top$ and boxed atoms by applying $\Diamond$ and $\wedge$.

A *simple Sahlqvist formula* is a modal formula of the form $\varphi \to \psi$, where φ is a simple Sahlqvist antecedent and ψ is a positive formula.

A *Sahlqvist antecedent* is built from $\bot, \top$, negative formulas and boxed atoms by applying $\Diamond$ and $\wedge$.

A *Sahlqvist implication* is a modal formula of the form $\varphi \to \psi$, where φ is a Sahlqvist antecedent and ψ is a positive formula.

A *Sahlqvist formula* is built from Sahlqvist implications by applying $\Box$ and $\vee$.

Theorem (Sahlqvist correspondence) For any Sahlqvist formula φ, there is a corresponding first-order sentence that holds in a frame iff φ is valid in the frame.

This sentence can be obtained from φ by a simple Sahlqvist-van Benthem algorithm. For simplicity we will consider only the case of simple Sahlqvist formulas.

Let φ be a simple Sahlqvist formula.

(1) Identify boxed atoms in the antecedent.

(2) Draw the picture that discusses the minimal valuation that makes the antecedent true. Name the worlds involved by $t_0, \ldots, t_n$.

(3) Work out the minimal valuation i.e., get a first-order expression for it in terms of the named worlds.

(4) Work out the standard translation of φ. Use the names you fixed for the variables that correspond to $\Diamond$'s in the antecedent.

(5) Pull out the quantifiers that bind t_i variables in the antecedent to the front. For this use the equivalences

$$\exists x \alpha(x) \wedge \beta \leftrightarrow \exists x(\alpha(x) \wedge \beta),$$

$$\exists x\alpha(x) \to \beta \leftrightarrow \forall x(\alpha(x) \to \beta).$$

(6) Replace all the predicates $P(x), Q(x)$, etc., with the first-order expression corresponding to the minimal valuation.

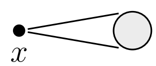
(7) Simplify, if possible.

(8) Add $\forall x$ (binding the free variable of the standard translation) to the resulting first-order formula to obtain the global first-order correspondent.

We will look at a few examples.

Let $\varphi = \Box p \to p$.

The diagram:



The minimal valuation is $V(p) = \{z : Rxz\}$.

The standard translation of φ is $\forall y(Rxy \to P(y)) \to P(x)$.

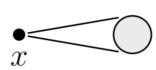
Replace $P(y)$ with Rxy and $P(x)$ with Rxx. We obtain $\forall y(Rxy \to Rxy) \to Rxx$.

This is equivalent to Rxx. By adding $\forall x$ we obtain the global first-order correspondent

$\forall x Rxx$ **reflexivity!**

Let $\varphi = \Box p \to \Box\Box p$.

The diagram:



The minimal valuation is $V(p) = \{z : Rxz\}$.

The standard translation of φ is

$$\forall y(Rxy \to P(y)) \to \forall z(Rxz \to \forall u(Rzu \to P(u)))$$

Replace $P(y)$ with Rxy and $P(u)$ with Rxu. We obtain

$$\forall y(Rxy \to Rxy) \to \forall z(Rxz \to \forall u(Rzu \to Rxu))$$

This is equivalent to

$$\forall z(Rxz \to \forall u(Rzu \to Rxu))$$

which is equivalent to

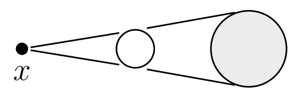
$$\forall z \forall u(Rxz \wedge Rzu \to Rxu)$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall z \forall u(Rxz \wedge Rzu \to Rxu) \quad \textbf{transitivity!}$$

Let $\varphi = \Box\Box p \to \Box p$.

The diagram:



The minimal valuation is $V(p) = \{z : \exists v(Rxv \wedge Rvz)\}$. The standard translation of φ is

$$\forall y(Rxy \to \forall z(Ryz \to P(z))) \to \forall u(Rxu \to P(u)))$$

Replace $P(u)$ with $\exists v(Rxv \wedge Rvu)$. In the antecedent we can replace $P(z)$ with the minimal valuation, but let us note that the instantiation of the standard translation of boxed atoms always gives us a tautology.

We obtain

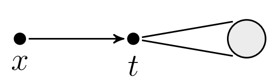
$$\forall u(Rxu \to \exists v(Rxv \wedge Rvu))$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall u(Rxu \to \exists v(Rxv \wedge Rvu)) \quad \textbf{density!}$$

Let $\varphi = \Diamond\Box p \to p$.

The diagram:



The minimal valuation is $V(p) = \{z : Rtz\}$.

The standard translation of φ is

$$\exists t(Rxt \wedge \forall z(Rtz \to P(z))) \to P(x)$$

Pull out the existential quantifier in the antecedent. We obtain

$$\forall t(Rxt \wedge \forall z(Rtz \to P(z)) \to P(x))$$

Replace $P(z)$ with Rtz and $P(x)$ with Rtx. We obtain

$$\forall t(Rxt \wedge \forall z(Rtz \to Rtz) \to Rtx)$$

This is equivalent to

$$\forall t(Rxt \to Rtx)$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall t(Rxt \to Rtx) \;\; \textbf{symmetry!}$$

Let $\varphi = p \to \Diamond p$.

The diagram:

$$\underset{x}{\bullet}$$

The minimal valuation is $V(p) = \{z : x = z\}$.

The standard translation of φ is

$$P(x) \to \exists y(Rty \wedge P(y))$$

Replace $P(y)$ with $x = y$ and note that the instantiation of the standard translation of boxed atoms is a tautology. We obtain

$$\exists y(Rxy \wedge y = x)$$

This is equivalent to

$$Rxx$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x Rxx \;\; \textbf{reflexivity!}$$

Let $\varphi = \Diamond\Diamond p \to \Diamond p$.

The diagram:

$$\begin{tikzcd} \underset{x}{\bullet} \arrow[r] & \underset{t_1}{\bullet} \arrow[r] & \underset{t_2}{\bullet} \end{tikzcd}$$

The minimal valuation is $V(p) = \{z : t_2 = z\}$.

The standard translation of φ is

$$\exists t_1(Rxt_1 \wedge \exists t_2(Rt_1t_2 \wedge P(t_2))) \to \exists y(Rxy \wedge P(y))$$

Pull out the existential quantifiers in the antecedent. We obtain

$$\forall t_1 \forall t_2((Rxt_1 \wedge Rt_1t_2 \wedge P(t_2)) \to \exists y(Rxy \wedge P(y)))$$

Replace $P(y)$ with $t_2 = y$ and note that the instantiation of the standard translation of boxed atoms is a tautology. We obtain

$$\forall t_1 \forall t_2((Rxt_1 \wedge Rt_1t_2) \to \exists y(Rxy \wedge (y = t_2)))$$

This is equivalent to

$$\forall t_1 \forall t_2((Rxt_1 \wedge Rt_1t_2) \to Rxt_2)$$

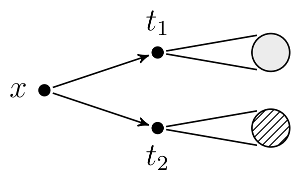
By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall t_1 \forall t_2((Rxt_1 \wedge Rt_1t_2) \to Rxt_2) \ \ \textbf{transitivity!}$$

If φ is a Sahlqvist formula, say $\Box(\varphi \to \psi) \vee \Box(\varphi' \to \psi')$ (where $\varphi \to \psi$ and $\varphi' \to \psi'$ are simple Sahlqvist formulas), then draw a diagram where outer $\Box$'s are treated as $\Diamond$'s of simple Sahlqvist formulas and $\vee$ is treated as $\wedge$ of simple Sahlqvist formulas.

Let $\varphi = \Box(\Box p \to q) \vee \Box(\Box q \to p))$.

The diagram:



The minimal valuation is $V(p) = \{z : Rt_1z\}$ and $V(q) = \{z : Rt_2z\}$.

The standard translation of φ (keeping in mind t_1 and t_2) is

$$\forall t_1(Rxt_1 \to (ST_{t_1}(\Box p) \to Q(t_1))) \vee \forall t_2(Rxt_2 \to (ST_{t_2}(\Box q) \to P(t_2)))$$

Pull out the quantifiers and replace $Q(t_1)$ with Rt_2t_1 and $P(t_2)$ with Rt_1t_2. Note again that the instantiation of the standard translation of boxed atoms gives a tautology.

We obtain

$$\forall t_1 \forall t_2((Rxt_1 \to Rt_2t_1) \vee (Rxt_2 \to Rt_2t_1))$$

which is equivalent to

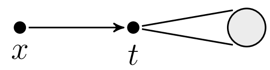
$$\forall t_1 \forall t_2((Rxt_1 \wedge Rxt_2) \to (Rt_1t_2 \vee Rt_2t_1))$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall t_1 \forall t_2((Rxt_1 \wedge Rxt_2) \to (Rt_1t_2 \vee Rt_2t_1)) \ \ \textbf{linearity!}$$

Let $\varphi = \Box(\Box p \to p)$.

The diagram:



The minimal valuation is $V(p) = \{z : Rtz\}$.

The standard translation of φ (keeping in mind t) is

$$\forall t(Rxt \to (ST_t(\Box p) \to P(t)))$$

Replace $P(t)$ with Rtt. Note again that the instantiation of the standard translation of boxed atoms gives a tautology.

We obtain

$$\forall t(Rxt \to Rtt)$$

By adding $\forall x$ we obtain the global first-order correspondent

$$\forall x \forall t(Rxt \to Rtt) \ \ \textbf{every successor is reflexive!}$$