Construction

Let $(\mathfrak{A}_i)_{i\in\mathbb{N}}$ be structures, let $\mathcal{U}$ be an ultrafilter, then define the ultraproduct $\mathfrak{U} := \prod_{i\in\mathbb{N}} \mathfrak{A}_i/\mathcal{U}$. For one fixed ultrafilter $\mathcal{U}_1$ write $\mathfrak{U}_{\mathcal{U}_1}$ instead.
Consider the definition of Interpretation of a model within a model, write "$\mathfrak{a}$" models that can be interpreted within $\mathfrak{A}$ (or the respective capital letter) and call "$I^{\mathfrak{A}}_{\mathfrak{a}^n}$" an interpretation function, for $\alpha \subseteq A^n$, $I^{\mathfrak{A}}_{\mathfrak{a}^n} : \alpha \to a$.
The symbol $\equiv$ is a relation between structures and is denotes elementary equivalence. It is a symbol of the metalanguage, though consider the following definition:

Definition ($\equiv^{\mathfrak{B}}$)**.** $\mathfrak{B} \models \mathfrak{b} \equiv \mathfrak{B} := \mathfrak{b} \equiv \mathfrak{B} + \exists_{n\in\mathbb{N}} \exists_{\beta\subseteq B^n} \exists_{I^{\mathfrak{B}}_{\mathfrak{b}^n}} I^{\mathfrak{B}}_{\mathfrak{b}^n}(\beta) = b$

Definition ($\mathfrak{U}^j$)**.** $\mathfrak{U}^j := \prod_{i\in\mathbb{N}\setminus\{j\}} \mathfrak{A}_i/\mathcal{U}$

Lemma (I.1)**.** $\exists_{\mathcal{U}} \forall_{j\in\mathbb{N}} \mathfrak{U} \equiv \mathfrak{U}^j_{\mathcal{U}}$

Proof. By Łos Theorem and for $\phi \in \mathcal{L}_\epsilon$, note $(\mathfrak{U} \models \varphi \Leftrightarrow \mathfrak{U}^j_{\mathcal{U}} \models \varphi) \Leftrightarrow (\{i \in \mathbb{N} : \mathfrak{A}_i \models \varphi\} \in \mathcal{U} \Leftrightarrow \{i \in \mathbb{N} \setminus \{j\} : \mathfrak{A}_i \models \varphi\} \in \mathcal{U})$, now divide two cases (a): $\mathfrak{A}_j \models \varphi$ and (b): $\mathfrak{A}_j \not\models \varphi$. Consider the latter and note "$\Leftarrow$" follows from closure under supersets. Now for $A \in \mathcal{U}$ and $x \in A$, assume for contradiction $A \setminus \{x\} \notin \mathcal{U}$, then $\bar{A} \cup \{x\} \in \mathcal{U}$, then $(\bar{A} \cup \{x\}) \cap A$, then $\{x\} \in \mathcal{U}$ and since $\mathcal{U}$ is non-principal on $\mathbb{N}$ and $|\{x\}| < \aleph_0$ get contradiction; hence $A \in \mathcal{U} \wedge x \in A \Rightarrow A \setminus \{x\} \in \mathcal{U}$, then "$\Rightarrow$" follows.
Now consider (a), $\mathfrak{A}_j \models \varphi$, and an ultrafilter containing $\mathbb{N} \setminus \{j\}$ and the claim follows trivially.

Theorem (I)**.** $\exists_{\mathfrak{u}} \mathfrak{U} \models \mathfrak{u} \equiv \mathfrak{U}$

Proof. By definition of "$\equiv^{\mathfrak{U}}$" and Łos Theorem, $\forall_{j\in\mathbb{N}}(\exists_{\mathfrak{a}_j}(\mathfrak{a}_j \equiv \mathfrak{U} + \exists_{n\in\mathbb{N}} \exists_{\alpha_j\subseteq U^n} \exists_{I^{\mathfrak{U}}_{\mathfrak{a}^n}} I^{\mathfrak{U}}_{\mathfrak{a}^n_j}(\alpha_j) = a_j)) \Leftrightarrow \{j \in \mathbb{N} : \mathfrak{A}_j \models \exists_{\mathfrak{a}_j} \mathfrak{a}_j \equiv \mathfrak{U}\} = \mathbb{N} \Rightarrow \mathfrak{U} \models \exists_{\mathfrak{u}} \mathfrak{u} \equiv \mathfrak{U}$. Hence consider two claims: first $\forall_{j\in\mathbb{N}} \exists_{\mathfrak{a}_j} \mathfrak{a}_j \equiv \mathfrak{U}$, which holds for $\mathfrak{a}_j = \mathfrak{U}_j$ by (I.1); second $\forall_{j\in\mathbb{N}}(\exists_{n\in\mathbb{N}} \exists_{\beta\subseteq U^n} \exists_{I^{\mathfrak{U}_j}_{\mathfrak{U}^n}} I^{\mathfrak{U}}_{\mathfrak{U}^n_j}(\beta) = U_j)$.

For $j \in \mathbb{N}$, consider for some $m \in \mathbb{N}$ and $\gamma \in A^m_j$, $I^{\mathfrak{U}_j}_{\mathfrak{A}^m_j} : \gamma \to U_j$
I have to find a way to define $\mathfrak{U}_j$ within $\mathfrak{A}_j$. An obvious way would be to use the fact that $\mathfrak{U}_j$ is an ultraproduct made by all models apart from $\mathfrak{A}_j$, though this would make $\mathfrak{A}_j$ have any arbitrary $\mathfrak{A}_i$ within itself and then, since I am ranging over all $j \in \mathbb{N}$, $\mathfrak{A}_i$ would have $\mathfrak{A}_j$ within itself, hence I implicitly assumed in the construction that $\mathfrak{A}_j$ contains itself, which was the claim I wanted to prove. I need instead to find a way to define $\mathfrak{U}_j$ that does not assume $\mathfrak{A}_j$ to contain itself.