Criteria of Identity and Structuralist Ontology

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Discussion Note: Criteria of Identity and Structuralist Ontology

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Abstract (100 words)

In discussions about whether the Principle of the Identity of Indiscernibles is compatible with structuralist ontologies of mathematics, it is usually assumed that individual objects are subject to criteria of identity which somehow account for the identity of the individuals. Much of this debate concerns structures that admit of non-trivial automorphisms. We consider cases from graph theory that violate even weak formulations of PII. We argue that (i) the identity or difference of places in a structure is not to be accounted for by anything other than the structure itself and that (ii) mathematical practice provides evidence for this view.

1. Introduction

There has been a good deal of discussion lately about whether the Principle of the Identity of Indiscernibles (PII) can be maintained in the context of structuralist ontologies of mathematics (Ladyman [2005], MacBride [2004, 2005, 2006], Button [2006]) and physics (Saunders [2006], MacBride [2005], Hawley [2006]). This debate is also about whether bona fide individual objects are subject to criteria of identity, construed as somehow accounting for the identity of the individual, and about whether or not relations can account for the numerical diversity of their relata. MacBride [2005] points out that Keranen [2001] presupposes that identity facts must be reducible to other kinds of facts and then suggests that identity facts might be primitive. Shapiro [2006, 2007] and Ketland [2006] explicitly reject the need to supply a general criterion of identity for places in structures. Button [2006] argues against Shapiro by questioning the existence of primitive identity facts (see our discussion below). Much of this literature is focused on examples of structures that admit of non-trivial automorphisms. For example, the complex plane admits of the non-trivial automorphism that maps every complex number to its complex conjugate, and the singlet state of two fermions admits of the non-trivial automorphism that permutes the two particles. In both cases the structure that results is invariant. The standard philosophical example of such a structure is that of Max Black's two qualitatively identical spheres that are a mile apart in empty space [1952]. Once again, permutation of the spheres results in the same structure. Ladyman [2005] and Saunders [2006] have pointed out that such examples do not violate PII in the weak form that involves only weak discernibility. All the above examples have in common that the entities involved stand in some irreflexive (but symmetric) relation to each other.

However, here we will consider cases in mathematics that fall foul of even this weak formulation of PII.[1] Whether or not there are corresponding cases in physics we do not know (though we will return to the case of empirical structures in the last part of this paper). Focusing on the former may highlight important differences between the domains of mathematics and physics. In any case, it is important that our discussion

[1] Button [2006] mentions similar examples, while drawing dissimilar conclusions from them; we will address his arguments in the last section. Cf. also Ketland's [2006], p.309, "dumb-bell" structure, which is used in order to argue against individuation by weak discernibility.

attempts to follow mathematical practice rather than any philosophical reconstruction of it. Though philosophical claims about mathematical objects cannot actually be *derived* from descriptions of mathematical practice, the more closely a position in the philosophy of mathematics resembles the ways in which mathematicians actually talk and reason, the more prima facie plausibility it has. The upshot of our examples below will be that (i) the identity or difference of places in a structure is not to be accounted for by anything other than the structure itself and that (ii) mathematical practice provides evidence that this is exactly the way in which mathematicians themselves conceive of places in structures. In order to justify these claims we will turn to graph theory. As we will show, if one wants to be faithful to the practice of graph theorists, one has to countenance graph-theoretic objects that are not even weakly discernible. We conclude that the identity relation for positions in a structure is a relation that ought to be viewed as an integral component of a structure in the same way as, for example, the successor relation is an integral component of the structure of natural numbers.

2. Graph Theory and Mathematical Structuralism

As mentioned before, in order to argue for this claim we find it instructive to consider a branch of mathematics that is in a sense elementary, has generated some of the most celebrated proofs and conjectures in twentieth century mathematics, and which yet has been mainly neglected in the discussions of mathematical structuralism so far, namely graph theory. Graphs are mathematical structures that contain only two kinds of objects, namely nodes and edges between nodes. Graphs may be undirected or directed depending on whether the relation between two nodes of their being joined by an edge is presupposed to be a symmetric relation. While graphs are "officially" defined set theoretically (as ordered pairs of a set of nodes and a set of edges, where edges are considered as ordered pairs of nodes in directed graphs and as unordered pairs of nodes in non-directed graphs), this set theoretic reconstruction does not preserve the manner in which graph theorists actually talk and reason about graphs. For a graph theorist, a graph presents itself as either being unlabelled or labelled: In an unlabelled graph, different nodes are indistinguishable if considered in isolation (which is why unlabelled graphs are of special interest to structuralists).[2] Labelled graphs are unlabelled graphs that come with an additional assignment of linguistic or numerical labels to their nodes, by which nodes become distinguishable by means of their labels even if taken in isolation.[3] While not

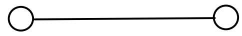
[2] Tutte [1984], p.6, who himself seems to take the position of a set-theoretic *in re* structuralist, acknowledges the *ante rem* inclinations of his fellow graph theorists, when he says: "Pure graph theory is concerned with those properties of graphs that are invariant under isomorphism, for example the number of vertices, the number of loops, the number of links, and the number of vertices of a given valency. It is therefore natural for a graph theorist to identify two graphs that are isomorphic. For example, all link-graphs are isomorphic, and therefore he speaks of the 'link-graph' as though there were only one. Similarly one hears of 'the null graph,' 'the vertex graph,' and 'the graph of the cube.' When this language is used, it is really an isomorphism class (also called an *abstract graph*) that is under discussion" (his emphasis).

[3] In the famous Black [1952] paper one of the two characters in the dialogue suggests distinguishing between the two globes by calling one 'Castor' and the other 'Pollux'; this case is directly analogous to the case of a labelled edgeless graph with order two. The other character points out that to use rather than to mention these names one must presuppose that some way of fixing their reference is available which begs the question at

being preserved properly in their set theoretic reconstruction, it is these intuitions on unlabelled and labelled graphs that are presupposed by graph theorists in their everyday work.[4]

Graphs may finally be called symmetric or asymmetric depending on whether or not there is a function that rearranges the nodes and leaves the graph structure unchanged, in other words, whether or not there is a non-trivial graph automorphism.[5]

Let us turn now to a few examples. The following unlabelled graph G with two nodes and one edge may be viewed as the graph theoretic counterpart of Black's two-spheres universe (or the field substructure consisting of the imaginary units *i* and –*i*, or the above mentioned singlet state of two fermions):



G is obviously symmetric. The two nodes in G are distinct, as can be concluded from the fact that they stand in the irreflexive (though symmetric) relation expressed by '*x* is connected to *y* by an edge (in G)'.

Next we apply a standard graph theoretic operation on G, namely, we "take away" the edge that relates the one node to the other.[6] The resulting graph G' looks as follows:[7]

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issue. We simply note that it is just assumed in graph theory that such labels can be deployed.

4 Brown [1999], p.106, quotes Alasdair Urquhart on this very point: "There is a mismatch between intuitive combinatorial mathematics and set theory. The set theory universe *imposes* a label on everything, so the labelled objects are primary by fiat. But this makes even elementary combinatorial mathematics clumsy. So it seems to me that such considerations throw some doubt on the 'mathematics = set theory' equation" (his emphasis).

5 Simple graphs, as we have characterised them above, can be further generalised to multigraphs, in which two nodes may be connected by more than one edge, or hypergraphs, where one edge may connect more than two nodes. Since these complications do not play any role for our argument, we omit any more detailed discussion. We also neglect the question of whether one allows for loops in graphs, i.e., edges that connect nodes to themselves.

6 It is worth noting that in an unlabelled graph with several nodes and no edges, if a node is removed, the same graph results regardless of which node is removed. Also if we have a graph with two nodes and one is removed and then a node is added there is no fact of the matter about which of the nodes in the graph we end up with is the one that was removed and replaced and which was left on its own in the intermediary graph of one node. This makes unlabelled graphs a good model for discussing issues concerning diachronic as well as synchronic identity in quantum mechanics (see, for example, the discussion in Lowe [2004]).

7 Shapiro [1997], p.115, considers patterns of "small cardinal numbers". His so-called 2-pattern can be regarded as corresponding to our graph G', however, while in his case the corresponding reference class is the collection of cardinal numbers, in our case the reference class of G' is the collection of unlabelled graphs. While the possible occurrence of edges is constitutive of the latter, edges do no play any role whatsoever for the former. Accordingly, Shapiro's 2-pattern could just as well be identified with G rather than G'.

For a graph theorist, G’ is just as much an unlabelled graph as G is. Indeed, a graph theorist’s answer to the question ‘how many unlabelled graphs with two nodes are there?’ is simple: ‘two’, i.e., G and G’. There is also nothing esoteric about this sort of question. In fact a great part of graph theory has been devoted to the enumeration of (classes of) unlabelled graphs, and a highly non-trivial mathematical theory – Polya’s enumeration theory – has been developed exactly for the purpose of answering this question.

Why is G’ now of particular interest as far as the above mentioned discussion about identity and structure is concerned? Since permuting the two nodes of G’ obviously leaves the graph unchanged, G’ allows for non-trivial automorphisms. Yet there is no irreflexive relation that may be used to ground the identity or difference of the nodes in accordance with the weak version of PII, nor is there any need of doing so; in the eyes of the graph theorist, the two nodes in G’ are perfectly respectable mathematical objects for which it is determinately true that they are distinct from each other. The fact that G’ consists of precisely two nodes is simply part of what G’ is, it is “built into” its graph theoretic structure. Adapting the structuralist slogan on natural numbers (as stated, for example, in MacBride [2005], p. 583), we are still allowed to say that “There is no more to the individual nodes ‘in themselves’ than the relations they bear to each other”, the only addition that we have to make is that we have to count identity and difference of nodes among the very relations that the nodes in a graph bear to each other. There is also no worry about whether we thereby ascribe any non-structural properties to unlabelled graphs: If the identity relation and the distinctness relation between nodes in a graph are logical relations, then the fact that a structure does or does not satisfy certain identity claims is just as structural as the fact that it has various properties which can be expressed by other logical concepts, such as existence, or by combinations thereof. On the other hand, if the identity relation for nodes in a graph is a genuinely mathematical relation, then – just as it is the case for every other mathematical relation – a structuralist ought to regard it as belonging to the very structure the places of which it does or does not relate. Finally, one should not misinterpret graphs as G’ to be in any sense exceptional or pathological: what has been said above applies equally well to all other unlabelled graphs that contain at least two isolated nodes (and thus to, e.g., 11 out of the 156 possible unlabelled graphs with 6 nodes). Furthermore, an analogous point can be made for all unlabelled graphs which include at least two distinct but isomorphic connected components, i.e., two isomorphic subgraphs for which there is no edge that leads from a node within one of the subgraphs to a node outside of it.
For example, the following graph G” is of this kind:

Although G” is again symmetric, there is no irreflexive relation, nor any other way of grounding the difference between these two components. It is a brute structural fact about this graph that it includes two components that are structurally indistinguishable if taken in isolation. If philosophers of mathematics tried to argue that either of G, G’, or G” was not a genuine mathematical structure that consisted of determinate components, they simply would not seem to get their data right. We conclude that the identity or difference of nodes or components in a graph – places in a structure – is not to be accounted for by anything else than the very graph itself, as this is the way in which graph theorists themselves conceive of unlabelled graphs.

3. Mathematics versus Metaphysics

MacBride [2005] argues that irreducible identity facts amount to “old news” because they turn structuralism into a form of Platonism. There are two kinds of Platonism that might be relevant here. MacBride is concerned with the claim that there are two separate categories of being, namely objects and relations. Keranen posits haecceities for individual mathematical objects and so defends a stronger form of Platonism. The latter we definitely deny since, because haecceities are intrinsic to each individual, the permutation of individuals always results in a new situation, whereas permuting exactly structurally similar individuals in a mathematical structure results in exactly the same structure. For example, permuting the two nodes in G’ above does not result in a new graph, and this is true even if the graph is labelled. We do not believe that accepting the existence of irreducible identity facts commits us to the weaker form of Platonism either. The latter does not however seem to be incompatible with *ante rem* structuralism and so we remain neutral on this issue.

Button [2006] has recently attacked the claim that there could be primitive identity facts of the kind discussed above. He gives two arguments against this view: (i) An epistemological one, according to which it is unclear how we could have epistemic access to primitive identity facts; (ii) A metaphysical one, which questions the “objecthood” of entities the numerical diversity of which would be primitive while they were at the same time indistinguishable from each other in terms of properties or relations. Let us review these two arguments against primitive identity facts in light of our previous discussion of unlabelled graphs.

How do we know that G’ consists of two nodes rather than just one? Certainly not by pointing to any property or relation by which the two nodes could be distinguished. But as posed the question is slightly misleading anyway. The actual question is: how do we know that there is an unlabelled graph with exactly two nodes and no edges? Once it is established that such a graph exists, then we can conclude that there is a unique such graph, since isomorphic structures are deemed to be identical according to the tenets of pattern structuralism. We would call this graph G’, and our knowledge of what G’ is like would be sufficient in order to know of every node x in G’ that there is another node y in G’ such that $x \neq y$ and that every node in G’ is either identical to x or y, i.e.: G’ consists of exactly two nodes. So the primary question to pose is the one about how we are able to know graphs as G’ to exist. But that question can be answered in more or less standard ways: because graph theory postulates it and we have every reason to believe that the basic principles of graph theory are coherent; because we can generate graphical templates that indicate so; and so forth. In particular, we can just consider the one node graph and apply the standard graph theoretic operation of adding a node, or we consider e.g. the two node directed graph and apply the operation of removing an edge. Hence, there is no real problem here, at least not over and above general epistemological worries about pattern structuralism the discussion of which would lead us far beyond the aims of this paper. (We think of questions such as: What does ‘coherent’ mean? How are we able to know structures on the basis of templates? Etc.)

As far as Button's second argument is concerned, the question is: Are nodes in symmetric unlabelled graphs proper objects, and if so, in what sense? For graph theorists, the answer is a simple one: nodes, edges, and unlabelled graph components are mathematical objects in the very same sense in which natural numbers or sets are; their objecthood is completely independent of whether they belong to a symmetric or to an asymmetric unlabelled graph. Graph theorists quantify over these objects, they consider identity claims for them, they count them, in short: they regard as them as objects proper. If this amounted in the eyes of the metaphysician to a merely "weak" logical or mathematical notion of object, then so be it – after all, why should the objecthood of mathematical objects conform to more than just logical or mathematical standards of *object*. Button argues that indiscernible mathematical objects are "unlike any objects with which [he] is familiar", and that they are "*metaphysically* suspicious" (his emphasis) (p. 220). We emphasise that there is no reason to expect mathematical objects (or quantum particles) to be like those with which we are familiar, and furthermore that the suspicions of metaphysicians weigh much less heavily with us than the implications of mathematical practice.

The remaining question is whether also physical entities could be objects in a similarly "weak" sense. In physics there are collections of fermions that admit of non-trivial automorphisms but there are irreflexive relations between their elements. The non-existence of such relations among collections of elementary bosons that admit of non-trivial automorphisms is taken as a reason for denying them the status of objects by Saunders. On the other hand, in mathematics it seems that there are structures that admit of non-trivial automorphisms where there are no irreflexive relations between their elements. This may be one important difference between mathematical and physical objects, or it may be taken as a reason for disputing the claim that elementary bosons are not objects. One philosopher who has applied graph theory to the metaphysics of physical reality in defence of a broadly structuralist view is Dipert [1997]. He claims that the world is an asymmetric graph because he believes that facts about the numerical identity and diversity of objects must supervene on the relational facts about each node in the graph representing the world's structure, whereas in symmetric graphs there are nodes that admit of exactly the same structure descriptions. We have argued that the only reason to accept such a supervenience requirement is the mistaken claim that facts about numerical identity and diversity must be grounded somehow; instead, the idea of a structure that does admit of non-trivial automorphism is perfectly intelligible and even suggested by mathematical practice. Carnap [1928], in his famous example of a railtrack network (cf. §14 of his *Aufbau*), acknowledges the possibility that the world might present itself as allowing for non-trivial automorphisms, in which case there would be two places in the network which would have to be regarded as indistinguishable by all scientific means. It remains an open question whether the empirical world has such a structure.[8]

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References

Black, M. 1952. The identity of indiscernibles. *Mind* 61: 153–64.
Brown, J.R. 1999. *Philosophy of Mathematics. An Introduction to the World of Proofs and Pictures*. London: Routledge.
Button, T. 2006. Realistic structuralism's identity crisis: a hybrid solution. *Analysis* 66: 216–22.
Carnap, R. 1928. *Der logische Aufbau der Welt*. Berlin: Weltkreis. Second edition, Hamburg: Meiner, 1961. Translated from the second edition as *The Logical Structure of the World.* Berkeley: University of California Press, 1967.
Dipert, R. 1997. The mathematical structure of the world: The world as graph. *Journal of Philosophy* XCIV: 329–58.
Hawley, K. 2006. Weak discernibility. *Analysis* 66: 300–3.
Ketland, J. 2006. Structuralism and the identity of indiscernibles. *Analysis* 66: 303–15.
Ladyman, J. 2005. Mathematical structuralism and the identity of indiscernibles. *Analysis* 65: 218–21.
Lowe, E.J. 2004. The four-category ontology: reply to Kistler. *Analysis* 64: 152–157.
MacBride, F. 2004. Introduction. In *The Philosophical Quarterly* 54: 1–15.
MacBride, F. 2005. Structuralism reconsidered. In *The Oxford Handbook of Logic and Mathematics*, ed. S. Shapiro. Oxford: Oxford University Press, pp. 563–89.
MacBride, F. 2006. What constitutes the numerical diversity of mathematical objects? *Analysis* 66: 63–69.
MacBride, F. (ed.) 2006. *Identity and Modality*. Oxford: Clarendon Press.
Saunders, S. 2006. Are quantum particles objects? *Analysis* 66: 52–63.
Shapiro, S. 1997. *Philosophy of Mathematics: Structure and Ontology*. Oxford: Oxford University Press.
Shapiro, S. 2006. Structure and identity & The governance of identity. In MacBride (2006), 109–145 & 164–173.
Shapiro, S. 2007. Identity, indiscernibility, and *ante rem* structuralism: the tale of *i* and *–i*. *Philosophia Mathematica*, this issue.
Tutte, W.T. 1984. *Graph Theory*. Menlo Park, California: Addison-Wesley.