Let $\{s_i \in \mathcal{L}_\epsilon : i \in \mathbb{N}\}$ s.t., for a theory T of $\mathcal{L}_\epsilon$:
(i): $\forall_{i\leq\mathbb{N}}(T \not\models s_i \wedge T \not\models \neg s_i)$
(ii): $\forall_{i,j\in\mathbb{N}}(i \neq j \Rightarrow T + s_i \models \neg s_j)$.
From (i) I derive that $\forall_{i\in\mathbb{N}} Con(T) \rightarrow Con(T + s_i)$, then choose a model of $T + s_i$ and call it $\mathfrak{A}_i$ s.t.
(iii): for a function f, $\mathfrak{A}_i \models f : \{\emptyset\} \rightarrow \mathbb{N} \wedge (f(\emptyset) = i \leftrightarrow s_i)$.
To clarify what I denote with (iii), I slit it into the two following cases, so that it's clear how $\mathfrak{A}_i$ can trivially be defined by induction given interpretations "$1^{\mathfrak{A}_i}$" and "$+^{\mathfrak{A}_i}$":
(iii.i): $\mathfrak{A}_1 \models f(\emptyset) = 1$
(iii.ii): $(\mathfrak{A}_i \models f(\emptyset) = i) \Rightarrow (\mathfrak{A}_{i+1} \models f(\emptyset) = i + 1)$
Note in "$\mathfrak{A}_i$" and "s_i", $i \in \mathbb{N}$ but $f(\emptyset) \in \mathbb{N}^{\mathfrak{U}_i} = \{n : n = 1^{\mathfrak{A}_i} +^{\mathfrak{A}_i} ... +^{\mathfrak{A}_i} 1^{\mathfrak{A}_i}\}$. Since $f(\emptyset)^{\mathfrak{A}_i} \in \mathbb{N}^{\mathfrak{A}_i}$ for $i = 1 + ... + 1 \in \mathbb{N}$, $\mathfrak{A}_i \models f(\emptyset) = 1 + ... + 1$.

Then, for a non-principal ultrafilter $\mathcal{U}$ on $\mathbb{N}$, define $\mathfrak{U} := \prod_{i\in\mathbb{N}} \mathfrak{A}_i/\mathcal{U}$, if not specified $\mathcal{U}$ is a free variable, if bounded then write "$\mathfrak{U}_\mathcal{U}$".
For $\varphi \in \mathcal{L}_\epsilon$ and the ultraproduct $\mathfrak{U}$ define $B_\mathfrak{U}(\varphi) := \{i \in \mathbb{N} : \mathfrak{A}_i \models \varphi\}$, note that this definition is independent from the ultrafilter $\mathcal{U}$.

Lemma (0). *For $\varphi \in \mathcal{L}_\epsilon$, $(\mathfrak{U} \models \varphi) \Rightarrow (|B_\mathfrak{U}(\varphi)| \geq \aleph_0)$*

Proof. Let $\varphi \in \mathcal{L}_\epsilon$ and assume $\mathfrak{U} \models \varphi$, by Łos Theorem derive that $\mathfrak{U} \models \varphi \Rightarrow \{i \in \mathbb{N} : \mathfrak{A}_i \models \varphi\} \subset \mathcal{U}$ and since $\mathcal{U}$ is non-principal derive $X \in \mathcal{U} \rightarrow |X| \geq \aleph_0$, hence $|\{i \in \mathbb{N} : \mathfrak{A}_i \models \varphi\}| \geq \aleph_0$.

Lemma (0'). *For $\varphi \in \mathcal{L}_\epsilon$, $|B_\mathfrak{U}(\varphi)| \geq \aleph_0 \Rightarrow \exists_\mathcal{U}(\mathfrak{U}_\mathcal{U} \models \varphi)$*

Proof. Let $\varphi \in \mathcal{L}_\epsilon$ and assume $|B_\mathfrak{U}(\varphi)| \geq \aleph_0$ then one can generate a non-principal ultrafilter on $\mathbb{N}$ from $B_\mathfrak{U}(\varphi)$, its existence proves the claim.

Theorem (I). $\mathfrak{U} \models \exists_{i\in\mathbb{N}}(f(\emptyset) = i)$

Proof. Note $B_\mathfrak{U}(\exists_{i\in\mathbb{N}}(f(\emptyset) = i)) = \{j \in \mathbb{N} : \mathfrak{A}_j \models \exists_{i\in\mathbb{N}}(f(\emptyset) = i)\} = \mathbb{N}$, the last step follows from (iii) and construction of $\mathfrak{A}_i$. Since $\mathcal{U}$ is an ultrafilter on $\mathbb{N}$ then $\mathbb{N} \in \mathcal{U}$ hence, by Łos Theorem, the claim follows.

Theorem (II). $\forall_{i\in\mathbb{N}}(\mathfrak{U} \models f(\emptyset) \neq i)$

Proof. Let $i \in \mathbb{N}$, then note by (ii) and (iii), $B_\mathfrak{U}(f(\emptyset) = i) = \{j \in \mathbb{N} : \mathfrak{A}_j \models f(\emptyset) = i\} = \{j \in \mathbb{N} : i = j\} = \{i\}$ since $\mathfrak{A}_i \models f(\emptyset) = j \Leftrightarrow i = j$ by (iii), also $|\{i\}| < \aleph_0$ therefore, by 0, $\mathfrak{U} \not\models f(\emptyset) = i$ and by completeness of models derive the claim.

Lemma. *For $\alpha^\mathfrak{U} \in \mathbb{N}^\mathfrak{U}$, $\mathfrak{U} \models f(\emptyset) = \alpha \Rightarrow \alpha > \omega$.*

Proof. From (I) define $\alpha^\mathfrak{U}$ s.t. $\mathfrak{U} \models f(\emptyset) = \alpha$. Assume $\exists_{i\in\mathbb{N}}\mathfrak{U} \models i = \alpha$ and derive contradiction by (II), hence the claim.

Theorem. $|\mathbb{N}^\mathfrak{U}| > \aleph_0$

Proof. Working on it. See theorem 3.12 of Models and Ultraproducts by Bell and Slomson.