Definition. $u := \bigcap_{x \in \mathcal{U}} x$

Definition. $F_{\mathcal{U}} = \{x \subseteq \omega : u \subseteq x\}$

Lemma 1. *$F_{\mathcal{U}}$ is a filter.*

Proof. Properties to check are: (a): $\emptyset \notin F_{\mathcal{U}}$, (b): $\forall_{X \in F_{\mathcal{U}}} \forall_{Y \subseteq \omega} X \subseteq Y \to Y \in F_{\mathcal{U}}$, (c): $\forall_{X,Y \in F_{\mathcal{U}}} X \cap Y \in F_{\mathcal{U}}$. (a), $u \not\subseteq \emptyset$, hence $\emptyset \notin u$. (b), let $X \in F_{\mathcal{U}}$ and $Y \subset \omega$ s.t. $X \subseteq Y$, then $u \subseteq Y$, hence $Y \in F_{\mathcal{U}}$. (c), let $X, Y \in F_{\mathcal{U}}$, then $u \subseteq X$ and $u \subseteq Y$, hence $u \subseteq X \cap Y$, and then $X \cap Y \in F_{\mathcal{U}}$.

Lemma 2. *$F_{\mathcal{U}}$ is not an ultrafilter*

Proof. Assume for contradiction that $F_{\mathcal{U}}$ were an ultrafilter, then it must hold (d): $\forall_{X \subseteq \omega} X \in F_{\mathcal{U}} \vee \overline{X} \in F_{\mathcal{U}}$. Consider $X \subseteq \omega$ s.t. $u \neq X \cap u \neq \emptyset$, then since $u \neq X \cap u$, $u \not\subseteq X$ hence $X \notin F_{\mathcal{U}}$. Since $X \cap u \neq \emptyset$, then $\overline{X} \cap u \neq u$ hence $u \not\subseteq \overline{X}$, and then $\overline{X} \notin F_{\mathcal{U}}$.

Lemma 3. *$F_{\mathcal{U}} \subsetneq \mathcal{U}$*

Proof. Let $X \in F_{\mathcal{U}}$, hence $u \subseteq X$ and since $u \in \mathcal{U}$ and $\mathcal{U}$ is closed under subset, then $X \in \mathcal{U}$. Since $\mathcal{U}$ is an ultrafilter, i.e. e maximal filter and $F_{\mathcal{U}}$ is not an ultrafilter, then $F_{\mathcal{U}} \neq \mathcal{U}$

Definition. *For $X \subseteq \mathcal{P}(\omega)$, $X^0 := X$, $X^{n+1} := \{x \subseteq \omega : \exists_{y,z \in X^n} y \cap z = x)\}$, $X^{cl_\cap} := X^\omega$*

Definition. $\mathcal{G}_{\mathcal{U}} := \{G \subseteq \mathcal{U} : \{x \subseteq \omega : (\exists_{y \in G^{cl_\cap}} y \subseteq x)\} = \mathcal{U}\}$

Hypothesis. $\forall_{G,H \in \mathcal{G}_{\mathcal{U}}} G \cap H \in \mathcal{G}_{\mathcal{U}}$

Proof. Let G, H be in $\mathcal{G}_{\mathcal{U}}$, I want to show that $\{x \subseteq \omega : (\exists_{y \in (G \cap H)^{cl_\cap}} y \subseteq x)\} = \mathcal{U}$.
I don't know. If that were the case, then $\mathcal{G}_{\mathcal{U}}$ would be a filter, perhaps ultrafilter!

Hypothesis. *$(\mathcal{G}_{\mathcal{U}}, \subsetneq)$ has a minimal element.*

Proof. Let C be a chain in $\mathcal{G}_{\mathcal{U}}$, then $\forall_{d \in C} \bigcap_{c \in C} c \subseteq d$, also $\bigcap_{c \in C} c \in \mathcal{G}_{\mathcal{U}}$ by Lemma (last Hypothesis) . Note that clearly $\mathcal{U} \in \mathcal{G}_{\mathcal{U}}$ then $\mathcal{G}_{\mathcal{U}} \neq \emptyset$. Conclude by Zorn's Lemma, the claim.

Definition. $\Gamma := \{\mathcal{G} \subseteq \mathcal{P}(\omega) : \exists_{\mathcal{U}} P(\mathcal{U}) \wedge \mathcal{G} = \mathcal{G}_{\mathcal{U}}\}$*, for P the property denoting being an ultrafilter.*

Definition. $\Gamma^! := \{\mathcal{G} \subseteq \mathcal{P}(\omega) : \exists!_{\mathcal{U}} P(\mathcal{U}) \wedge \mathcal{G} = \mathcal{G}_{\mathcal{U}}\}$

Definition. $\dot{\Gamma} := \{x \subseteq \omega : \exists_{y \in \Gamma} \exists_{z \in y} z = x\}$

Hypothesis. $\Gamma = \Gamma^!$

Hypothesis. $\forall_{x,y \in \dot{\Gamma}} x \cap y \in \dot{\Gamma}$

Hypothesis. $\forall_{x \in \dot{\Gamma}} \overline{x} \in \dot{\Gamma}$